PHYSICAL REVIEW E **78**, 027601 (2008)

Eigenvalue cutoff in the cubic-quintic nonlinear Schrödinger equation

Vladyslav Prytula,[1] Vadym Vekslerchik,[1,2] and Víctor M. Pérez-García[1]

[1]*Departamento de Matemáticas, E.T.S. Ingenieros Industriales and Instituto de Matemática Aplicada a la Ciencia y la Ingeniería, Universidad de Castilla–La Mancha, Avenida Camilo José Cela 3, Ciudad Real, 13071 Spain*

[2]*A. Ya. Usikov Institute for Radiophysics and Electronics, National Academy of Sciences of Ukraine, 12 Proskura Street, 61085 Kharkov, Ukraine*

(Received 2 July 2008; published 25 August 2008)

Using theoretical arguments, we prove the numerically well-known fact that the eigenvalues of all localized stationary solutions of the cubic-quintic $(2+1)$-dimensional nonlinear Schrödinger equation exhibit an upper cutoff value. The existence of the cutoff is inferred using Gagliardo-Nirenberg and Hölder inequalities together with Pohozaev identities. We also show that, in the limit of eigenvalues close to zero, the eigenstates of the cubic-quintic nonlinear Schrödinger equation behave similarly to those of the cubic nonlinear Schrödinger equation.

DOI: 10.1103/PhysRevE.78.027601 PACS number(s): 05.45.Yv, 42.65.Tg

I. INTRODUCTION

The nonlinear Schrödinger equation (NLSE) has been widely used in modeling nonlinear wave dynamics in many physical scenarios, such as nonlinear optics [1,2], plasma physics [3], Bose-Einstein condensates [4], biomolecular dynamics [5], and others [6–8]. The simplest scalar NLSE is of the form

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + f(|\psi|^2)\psi, \tag{1}$$

where ψ is a complex field defined usually on the whole $\mathbb{R}^n$ with $n=1,2,3$ and f describes the nonlinear response of the medium. Many different types of nonlinearities $f(|\psi|^2)$ arise in the different physical fields of applicability of the equation including power-law, saturable, and nonlocal nonlinearities to cite a few examples. The most relevant one, for which many theoretical and analytical studies of NLSEs have been done, is the classical cubic nonlinearity $f(|\psi|^2)=g|\psi|^2$, both because of its direct interest and also because it corresponds to the simplest nonlinear response proportional to the square of the involved field [namely, the light intensity in optics, the number of particles in Bose-Einstein condensation (BEC) applications, etc.].

One of the simplest extensions of the cubic NLSE is the so-called cubic-quintic NLS (CQNLS) model, which, in normalized units, is

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi - g|\psi|^2\psi + h|\psi|^4\psi. \tag{2}$$

The CQNLS equation is another universal mathematical model describing many situations of physical interest and approximating other more complicated ones. As examples, it arises in plasma physics [9,10], condensed matter physics [11], nuclear physics [12], Bose-Einstein condensation [13], etc., but probably the application of the model which has attracted more attention in the last years is the description of the propagation of paraxial beams in certain nonlinear optical media. Many different optical materials have a refractive index that can be well described by a cubic-quintic nonlinearity such as some semiconductors and doped glasses (e.g., AlGaAs [14] and $\mathrm{CdS}_x\mathrm{Se}_{1-x}$ [15]), the polydiacetylene paratoluene sulfonate (PTS) [16], chalcogenide glasses [17], some transparent organic materials [18], or even media with complex susceptibilities induced by electromagnetically induced transparency [19].

We will consider localized stationary solutions (i.e., solitary waves or solitons) of Eq. (2). Thus, taking $\psi(x,t)=u(x)e^{i\beta t}$ we will study solutions of

$$\beta u = \Delta u + (g|u|^2 - h|u|^4)u, \tag{3}$$

with u a complex function defined on $\mathbb{R}^2$ and vanishing at infinity. The stationary solutions of Eq. (3) and their stability properties have been studied in many papers [10,20–35].

A well-known fact is that no solitary waves are found beyond a maximum value of the eigenvalue $\beta=\beta_*$. In fact, for $\beta\leq\beta_*$ one finds many solutions with different widths corresponding to very different values of the norm of the solution $N_\beta=\int_{\mathbb{R}^2}|u_\beta|^2dx$. In fact, when $\beta\to\beta_*$, $N_\beta\to\infty$. Thus, there is a cutoff in the permitted eigenvalue of localized solutions of the CQNLS model. The existence of this cutoff and its specific value as a function of the parameters was discussed in many papers. For fundamental states it was studied numerically [36–38] and by approximate variational methods with a super-Gaussian ansatz in Ref. [23] and also in Ref. [28]. For vortex states the same problem was studied by means of variational [33], numerical [33], and analytical methods [26,31].

In this paper we provide theoretical support for previous works dealing with the problem of the cutoff [23,26,28,31,33,36] and prove in a rigorous yet simple way that localized stationary states of the cubic-quintic NLSE exist only for eigenvalues on a finite interval. The obtained result holds for all types of localized solutions—e.g., ground, vortex, or dipole states—of the cubic-quintic NLSE.

The article is organized as follows: first in Sec. II we introduce the NLSE to be considered, and the integral quantities and the inequalities to be used in the subsequent sections. Section III contains the derivation of the upper limit for the eigenvalues and analysis of the behavior near the cutoff. Next, in Sec. IV we describe the related problem of the asymptotic behavior of the stationary states in the NLSE limit. Finally, in Sec. V we summarize our results.

II. STATEMENT OF THE PROBLEM

The typical situation for the cubic-quintic nonlinearity corresponds to the case where we have a combination of focusing cubic and defocusing quintic terms. The interplay of focusing and defocusing nonlinearities in that case prevents the wave collapse [38] and is responsible for the liquidlike features of the localized stationary states [21,28,33].

In this section we will consider the case of two spatial dimensions, which is the one relevant to nonlinear optics and many other applications of the model.

The equation we are dealing with can be written as

$$\beta u = \Delta u + g u^3 - h u^5, \tag{4}$$

with $\beta, g, h > 0$. Our analysis will use the energy

$$K = -\beta N + g U_2 - h U_3 \tag{5}$$

and Pohozaev identities [39,40]

$$0 = \beta N - \frac{g}{2} U_2 + \frac{h}{3} U_3, \tag{6}$$

where the moments of u are defined by

$$K = \int_{\mathbb{R}^n} d^n\mathbf{r} |\boldsymbol{\nabla} u|^2, \tag{7a}$$

$$N = \int_{\mathbb{R}^n} d^n\mathbf{r}\, u^2, \tag{7b}$$

$$U_j = \int_{\mathbb{R}^n} d^n\mathbf{r}\, u^{2j}, \quad j = 2,3. \tag{7c}$$

III. CUTOFF IN THE CUBIC-QUINTIC NLSE

A. Existence of a cutoff

By direct calculations from (5) and (6) we get the following *bilinear* relation between the moments:

$$4hK^2 + (3g^2 - 16\beta h)NK + \beta(16\beta h - 3g^2)N^2 + g^2 h(NU_3 - U_2^2) = 0. \tag{8}$$

This relation can be rewritten in terms of $X = N/K$:

$$\beta(\beta - \beta_*)X^2 + (\beta_* - \beta)X + \frac{1}{4}(1 + \Gamma) = 0, \tag{9}$$

where

$$\Gamma = \frac{g^2}{4}\frac{NU_3 - U_2^2}{K^2}, \quad \beta_* = \frac{3}{16}\frac{g^2}{h}. \tag{10}$$

It should be noted that it follows from the Hölder inequality

$$NU_3 \geqslant U_2^2 \tag{11}$$

that $\Gamma > 0$. Now, since our quadratic equation (9) must have real roots, after calculating its discriminant

$$\mathcal{D} = (\beta_* - \beta)(\beta_* + \beta\Gamma), \tag{12}$$

one can conclude that

$$\beta < \beta_*. \tag{13}$$

Thus, there is an upper bound on the eigenvalues $3g^2/16h$ and localized stationary states of Eq. (4) exist only when β lies within the interval

$$\beta \in \left(0, \frac{3}{16}\frac{g^2}{h}\right). \tag{14}$$

This result agrees with previous numerical and approximate calculations for this quantity [23,26,28,31,33,36], but here is obtained rigorously using simple arguments (see also Ref. [41]). In fact, the comparison with the numerical results of Refs. [23,28,33] shows that our bound is optimal and matches closely the numerics.

B. Behavior near the cutoff

A natural question that arises, after establishing the bound (13), is what occurs with the solution when the parameter β approaches the cutoff limit. The aim of this section is to obtain a bound for the quantity N which allows us to show that $N \to +\infty$ when $\beta \to \beta_*$.

Solving Eq. (9) we obtain that

$$\frac{N}{K} = \frac{1}{2\beta}\left(1 + \sqrt{\frac{\beta_* + \Gamma\beta}{\beta_* - \beta}}\right). \tag{15}$$

The sign before the radical is determined by the identity

$$2\beta N - K = \frac{1}{2} g U_2 > 0, \tag{16}$$

which is a direct consequence of Eqs. (5) and (6). Since $\Gamma > 0$, Eq. (15) leads to the inequality

$$\frac{N}{K} > \frac{1}{2\beta}\sqrt{\frac{\beta_*}{\beta_* - \beta}}. \tag{17}$$

On the other hand, we can obtain another type of bounds for K and N. From Eqs. (5) and (6) we get

$$\frac{h}{3} U_3 = \beta N - K, \tag{18a}$$

$$K = \frac{1}{2} g U_2 - \frac{2}{3} h U_3. \tag{18b}$$

Combining these relations and the identity

$$\frac{\partial K}{\partial \beta} = N, \tag{19}$$

which can be derived by differentiating Eq. (4) and applying again Eqs. (5) and (6), we conclude that K is a monotone function. These lead to the fact that

$$K = 0 \quad \text{when } \beta = 0 \tag{20}$$

and, respectively, $K=\int_0^\beta N d\beta$. Now let us recall the particular form of the Gagliardo-Nirenberg [42–46] inequality, which in our case can be written as

$$U_2 \leqslant C_{GN} K N, \tag{21}$$

where C_{GN} is the optimal constant for the Gagliardo-Nirenberg inequality in two dimensions. Applying this inequality to Eq. (18b) we get

$$N \geqslant \frac{2}{C_{GN} g}. \tag{22}$$

Combining Eqs. (22) and (19) we get the inequality

$$K \geqslant \frac{2}{C_{GN} g}\beta. \tag{23}$$

Substituting this estimate into (17) we obtain finally

$$N \geqslant \frac{1}{C_{GN}}\sqrt{\frac{\beta_*}{\beta_* - \beta}}, \tag{24}$$

which guarantees that $N \to +\infty$ as $\beta \to \beta_* \equiv 3g^2/(16h)$. Thus we can see that the cutoff phenomenon is a manifestation of the blowup of the norm (note that N is nothing but the L^2 norm of u).

Another consequence of the estimate (24) is the fact that N is bounded from below by $1/C_{GN}$ when $\beta \to 0$.

IV. LIMIT $\beta \to 0$

It is possible to analyze the asymptotic behavior of solutions of Eq. (4) when $\beta \to 0$. To do it we use the rescaling symmetry of our equation; specifically, we will use the fact that if u is a solution of Eq. (4), then the function $\tilde{u}$ given by $\tilde{u}(\mathbf{r}) = \xi u(\eta \mathbf{r})$ solves

$$\tilde{\beta} u = \Delta \tilde{u} + \tilde{g}\tilde{u}^3 - \tilde{h}\tilde{u}^5, \tag{25}$$

where

$$\tilde{\beta} = \eta^2 \beta, \quad \tilde{g} = \frac{\eta^2}{\xi^2} g, \quad \tilde{h} = \frac{\eta^2}{\xi^4} h. \tag{26}$$

Calculating the moments of the function $\tilde{u}$ we arrive at the following rescaling relation for K and N treated as functions of the parameters β, g, and h:

$$K(\beta, g, h) = \frac{1}{\xi^2} K\left(\eta^2\beta, \frac{\eta^2}{\xi^2} g, \frac{\eta^2}{\xi^4} h\right), \tag{27a}$$

$$N(\beta, g, h) = \frac{\eta^2}{\xi^2} N\left(\eta^2\beta, \frac{\eta^2}{\xi^2} g, \frac{\eta^2}{\xi^4} h\right). \tag{27b}$$

Further, choosing

$$\eta^2 = \frac{1}{\beta}, \quad \xi^2 = \frac{g}{\beta},$$

we can obtain

$$K(\beta, g, h) = \frac{\beta}{g} K\left(1, 1, \frac{\beta h}{g^2}\right), \tag{28a}$$

$$N(\beta, g, h) = \frac{1}{g} N\left(1, 1, \frac{\beta h}{g^2}\right). \tag{28b}$$

Considering the limit $\beta \to 0$ we find the following asymptotic formulas for $N[u]$ and $K[u]$:

$$N[u] = \frac{1}{g} N[\phi] + O(\beta), \tag{29a}$$

$$K[u] = \frac{\beta}{g} K[\phi] + \frac{\beta^2 h}{3g^3} U_3[\phi] + O(\beta^3), \tag{29b}$$

where ϕ is the Townes soliton—i.e., the solution of

$$\Delta\phi = \phi - \phi^3. \tag{30}$$

These results make it possible to deduce that in the asymptotic region as $\beta \to 0$ the behavior of solutions of the cubic-quintic Schrödinger equation is similar to the behavior of eigensolutions of the cubic nonlinear Schrödinger equation.

V. CONCLUSIONS

Using general mathematical arguments we have shown that the localized stationary solutions of the cubic-quintic nonlinear Schrödinger equation exhibit an eigenvalue cutoff. This result holds for all types of localized stationary solutions—e.g., fundamental, vortex, or dipole states. The cutoff is in excellent agreement with previous approximate or numerical results. We have also obtained a lower bound for the number of particles in the eigenstates that diverges exactly at the cutoff $\beta = \beta_*$. Finally, in the limit of small eigenvalues we obtain the result that the eigenstates of the cubic-quintic nonlinear Schrödinger equation behave in the same way as those of the cubic nonlinear Schrödinger equation.

Our results complement present knowledge on one of the key models of mathematical physics and support in a rigorous yet simple way previous numerical observations.

ACKNOWLEDGMENTS

This work has been partially supported by Ministerio de Educación y Ciencia (MEC), Spain (Project No. FIS2006-04190) and Junta de Comunidades de Castilla–La Mancha (Project No. PCI-08-0093). V.P. acknowledges support from MEC through Grant No. AP2005-4528.

[1] Y. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic, San Diego, 2003).
[2] A. Hasegawa, *Optical Solitons in Fibers* (Springer-Verlag, Berlin, 1989).
[3] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic, New York, 1982).
[4] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[5] A. S. Davydov, *Solitons in Molecular Systems* (Reidel, Dordrecht, 1985).
[6] C. Sulem and P. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse* (Springer, Berlin, 2000).
[7] *Nonlinear Klein-Gordon and Schrödinger Systems: Theory and Applications*, edited by L. Vázquez, L. Streit, and V. M. Pérez-García (World Scientific, Singapore, 1997).
[8] Kh. I. Pushkarov, D. I. Pushkarov, and I. V. Tomov, Opt. Quantum Electron. **11**, 471 (1979); Kh. I. Pushkarov and D. I. Pushkarov, Rep. Math. Phys. **17**, 37 (1980).
[9] C. T. Zhou and X. T. He, Phys. Scr. **50**, 415 (1994).
[10] T. A. Davydova, A. I. Yakimenko, and Yu. A. Zaliznyak, Phys. Rev. E **67**, 026402 (2003).
[11] I. V. Barashenkov and V. G. Makhankov, Phys. Lett. A **128**, 52 (1988); I. V. Barashenkov and E. Yu Panova, Physica D **69**, 114 (1993).
[12] V. G. Kartavenko, Sov. J. Nucl. Phys. **40**, 240 (1984).
[13] A. Bulgac, Phys. Rev. Lett. **89**, 050402 (2002); F. Kh. Abdullaev, A. Gammal, L. Tomio, and T. Frederico, Phys. Rev. A **63**, 043604 (2001); A.-X. Zhang and J.-K. Xue, *ibid.* **75**, 013624 (2007).
[14] S. Tanev and D. I. Pushkarov, Opt. Commun. **141**, 322 (1997).
[15] L. H. Acioli, A. S. L. Gomes, J. M. Hickmann, and C. B. de Araujo, Appl. Phys. Lett. **56**, 2279 (1990); P. Roussignol, D. Ricard, J. Lukasik, and C. Flytzanis, J. Opt. Soc. Am. B **4**, 5 (1987).
[16] B. L. Lawrence and G. I. Stegeman, Opt. Lett. **23**, 591 (1998).
[17] K. Ogusu, J. Yamasaki, S. Maeda, M. Kitao, and M. Minakata, Opt. Lett. **29**, 265 (2004).
[18] C. Zhan, D. Zhang, D. Zhu, D. Wang, Y. Li, Z. Lu, L. Zhao, and Y. Nie, J. Opt. Soc. Am. B **19**, 369 (2002).
[19] H. Michinel, M. J. Paz-Alonso, and V. M. Perez-Garcia, Phys. Rev. Lett. **96**, 023903 (2006).
[20] M. Quiroga-Teixeiro and H. Michinel, J. Opt. Soc. Am. B **14**, 2004 (1997).
[21] C. Josserand and S. Rica, Phys. Rev. Lett. **78**, 1215 (1997).
[22] D. V. Skryabin and W. J. Firth, Phys. Rev. E **58**, 3916 (1998).
[23] K. Dimitrevski, E. Reimhult, E. Svensson, A. Öhgren, D. Anderson, A. Berntson, M. Lisak, and M. L. Quiroga-Teixeiro, Phys. Lett. A **248**, 369 (1998).
[24] J. Yang and D. J. Kaup, SIAM J. Appl. Math. **60**, 967 (2000).
[25] I. Towers, A. V. Buryak, R. A. Sammut, B. A. Malomed, L.-C. Crasovan, and D. Mihalache, Phys. Lett. A **288**, 292 (2001).
[26] V. I. Berezhiani, V. Skarka, and N. B. Aleksić, Phys. Rev. E **64**, 057601 (2001).
[27] Z. Jovanovski, J. Mod. Opt. **48**, 865 (2001).
[28] H. Michinel, J. Campo-Taboas, R. García-Fernández, J. R. Salgueiro, and M. L. Quiroga-Teixeiro, Phys. Rev. E **65**, 066604 (2002).
[29] B. A. Malomed, L. C. Crasovan, and D. Mihalache, Physica D **161**, 187 (2002).
[30] D. Mihalache, D. Mazilu, L.-C. Crasovan, I. Towers, A. V. Buryak, B. A. Malomed, L. Torner, J. P. Torres, and F. Lederer, Phys. Rev. Lett. **88**, 073902 (2002).
[31] R. L. Pego and H. A. Warchall, J. Nonlinear Sci. **12**, 347 (2002).
[32] D. Mihalache, D. Mazilu, I. Towers, B. A. Malomed, and F. Lederer, Phys. Rev. E **67**, 056608 (2003).
[33] H. Michinel, J. R. Salgueiro, and M. J. Paz-Alonso, Phys. Rev. E **70**, 066605 (2004).
[34] T. A. Davydova and A. I. Yakimenko, J. Opt. A, Pure Appl. Opt. **6**, S197 (2004).
[35] A. S. Desyatnikov, D. Mihalache, D. Mazilu, B. A. Malomed, C. Denz, and F. Lederer, Phys. Rev. E **71**, 026615 (2005).
[36] D. Anderson and G. Derrick, J. Math. Phys. **11**, 1336 (1970); **12**, 945 (1971).
[37] D. Mihalache, D. Mazilu, F. Lederer, Y. V. Kartashov, L.-C. Crasovan, L. Torner, and B. A. Malomed, Phys. Rev. Lett. **97**, 073904 (2006).
[38] N. E. Kosmatov, V. F. Shvets, and V. E. Zaharov, Physica D **52**, 19 (1991).
[39] S. I. Pohozaev, Sov. Math. Dokl. **5**, 1408 (1965).
[40] H. Berestycki and P. L. Lions, Arch. Ration. Mech. Anal. **82**, 313 (1983).
[41] W. A. Strauss, Commun. Math. Phys. **55**, 149 (1977).
[42] M. Del Pino and J. Dolbeault, J. Math. Pures Appl. **9**, 847 (2002).
[43] T. Cazenave, *Introduction to Nonlinear Schrödinger Equations* (Textos de Métodos Matemáticos, Rio de Janeiro, 1989).
[44] M. I. Weinstein, Commun. Math. Phys. **87**, 567 (1983).
[45] L. Nirenberg, Ann. Sci. Ec. Normale Super. **13**, 116 (1959).
[46] E. Gagliardo, Ric. Mat. **7**, 102 (1958).